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AN EULERIAN METHOD FOR CALCULATING STRENGTH DEPENDENT DEFORMATION

PART ONE

Work done by:

J. K. Dienes
M. W. Evans
L. J. Hageman
W. E. Johnson
J. M. Walsh

Report written by:

J. K. Dienes
M. W. Evans
L. J. Hageman
W. E. Johnson
J. M. Walsh

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J. K. Dienes, M. W. Evans, L. J. Hageman,
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PART ONE

A DERIVATION FOR THE FLOW EQUATIONS FOR STRENGTH DEPENDENT DEFORMATION

by

J. K. Dienes

1. INTRODUCTION

Equations for the motion of a continuous medium capable of supporting shear stresses are reported by a number of authors,[1][2] but the complete set does not appear to have been published in a form suitable for solution by Eulerian hydrodynamic codes. Specifically, the Eulerian equations of motion for a compressible medium acted upon by a general stress tensor are required. In this volume the equations of motion are discussed starting from the principles of mass, momentum and energy conservation for finite masses. From these the differential equations of motion are derived. The constitutive equations relating stress and strain are required to complete the mathematical description.

It is also shown that the difference equations for hydrodynamic codes can be conveniently obtained from the integral form of the conservation laws. This is more convenient for a particular coordinate system than using the general method of tensor analysis and covariant differentiation, which tends to be awkward when deriving equations of motion in a specific curvilinear coordinate system.

The nature of a medium is specified by its equation of state, which is used to calculate the pressure from the density and specific internal energy, and a tensor constitutive equation relating deviator stresses, deviator strains and their rates. Constitutive equations for the shear stresses are discussed and the appropriate forms for elastic, elastic-plastic and rigid-plastic solids as well as for viscous fluids are presented.

2. A USEFUL RELATION

Before proceeding to discuss the specific conservation laws, it will prove useful to formulate an integral relation connecting the Eulerian and Lagrangian descriptions of fluid motion. This relation

$$\frac{\partial}{\partial t} \int_{\tilde{V}} H dv = \frac{D}{Dt} \int_{\bar{V}} H dv - \int_{\bar{S}} u_i n_i \, H ds \tag{1}$$

expresses the physical notion that the rate of change of an integral over a volume fixed in space ($\tilde{V}$) is the difference of two terms, a Lagrangian derivative (taken with respect to the moving element of mass contained in $\tilde{V}$) and a rate associated with mass transport out of $\tilde{V}$.

To prove this relation consider the volume integral

$$J(t) = \int_{\bar{V}} H(t) \, dv$$

over the volume $\bar{V}$ containing a fixed element of mass. In an interval Δt the change of J can be written

$$\Delta J = \int_{\bar{V}(t+\Delta t)} H(t+\Delta t) \, dv - \int_{\bar{V}(t)} H(t) \, dv$$

which, to the first order in Δt, can be expressed as

$$\Delta J = \int_{\bar{V}(t+\Delta t)-\bar{V}(t)} H(t) \, dv + \Delta t \int_{\bar{V}(t)} \dot{H}(t) \, dv .$$

By reference to Fig. 1 (where $\bar{S}$ is the surface bounding $\bar{V}$, n_i denotes the i component of the outward drawn normal, and the u_i are the components of the velocity vector) it can be observed that the thickness of the volume swept out by $\bar{S}$ in time Δt is $u_i n_i \, \Delta t$.*

*Tensor notation is used throughout this report, so that in this expression summation over repeated indices is to be understood. The text by Brillouin(2) gives a good discussion of the physics and mathematics appropriate here.

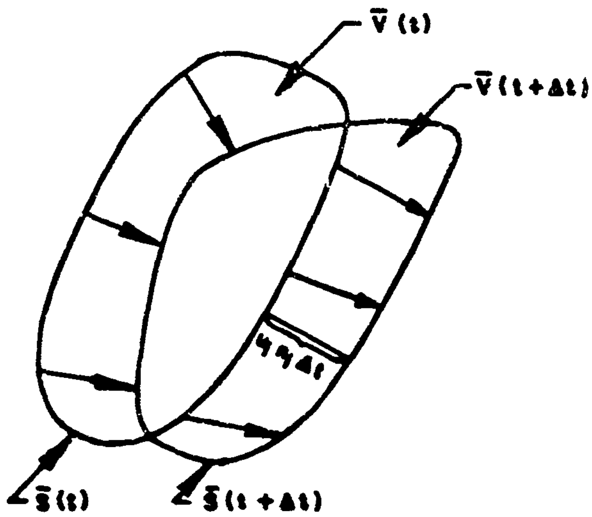


Fig. 1--Illustrating the motion of the volume $\overline{V}$ and the volume it sweeps out

Then the expression above for ΔJ becomes

$$\Delta J = \int_{\overline{S}} H(t)(u_i n_i \Delta t)\, dS + \Delta t \int_{\overline{V}} \dot{H}(t)\, dv \quad .$$

If, in the integral over $\overline{V}$ shown above, $\overline{V}$ is replaced by its instantaneous counterpart, $\widetilde{V}$, which is fixed in space, then the time derivative can be taken outside the integral. When that is done and the resulting equation is divided by Δt, the expression for the change of J which was written as Eq. 1 is obtained:

$$\frac{\partial}{\partial t}\int_{\widetilde{V}} H(t)\, dv = \frac{D}{Dt}\int_{\overline{V}} H(t)\, dv - \int_{\overline{S}} u_i n_i\, H dS \quad .$$

Here

$$\frac{DJ}{Dt} = \lim_{\Delta t \to 0} \frac{\Delta J}{\Delta t}$$

denotes the derivative of J as the mass in $\overline{V}$ moves along with the fluid. Special cases of interest are those in which the integral on the left of Eq. 1 denotes the mass, momentum and energy within a fixed element of volume, and will be discussed in the next paragraphs.

3. CONSERVATION OF MASS

If we put for H the density, ρ, of the fluid, then the first integral on the right of Eq. 1 becomes the change of mass in $\overline{V}$, which vanishes since $\overline{V}$ is defined to contain a fixed element of mass. Thus we find

$$\frac{\partial}{\partial t}\int_{\tilde{V}} \rho \, dv = -\int_{\tilde{S}} \rho \, u_i n_i \, dv \qquad (2)$$

and applying Gauss' theorem

$$\int_S F_i \, n_i \, dS = \int_V F_{i,i} \, dv \qquad (3)$$

we find

$$\int_{\tilde{V}} [\dot{\rho} + (\rho u_i)_{,i}] \, dv = 0$$

and, since the volume $\tilde{V}$ is arbitrary,

$$\dot{\rho} + (\rho u_i)_{,i} = 0 \qquad (4)$$

A compact notation is obtained with the convective derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum u_i \frac{\partial}{\partial x_i} \qquad (5)$$

The final form of the mass equation is, in the current notation:

$$\frac{D\rho}{Dt} + \rho u_{i,i} = 0 \qquad (6)$$

4. CONSERVATION OF MOMENTUM

To calculate the momentum of a finite volume we require unit vectors having a fixed direction, D, in space. The components of these vectors may, of course, vary from point to point in a curvilinear coordinate system.

Letting b_i denote the components of such a vector, the D-component of momentum of the mass in $\tilde{V}$ is given by

$$\int_{\tilde{V}} \rho u_i \, b_i \, dv$$

The rate of change of momentum, by Newton's law, is equal to the component in the fixed direction, D, of the surface tractions over $\tilde{S}$, i.e.

$$\frac{D}{Dt}\int_{\tilde{V}} \rho u_i \, b_i \, dv = \int_{\tilde{S}} \sigma_{ij} \, b_i \, n_j \, dS$$

where σ_{ij} is the total stress tensor, the force per unit area in $\tilde{S}$ is $\sigma_{ij} \, n_j = F_i$ and the component of force in the direction of b_i is $F_i \, b_i = \sigma_{ij} \, n_j \, b_i$. The general relation of Eq. 1 then becomes

$$\frac{\partial}{\partial t}\int_{\tilde{V}} \rho u_i \, b_i \, dv = \int_{\tilde{S}} T_{ij} \, n_j \, b_i \, dS \tag{7}$$

where the two integrals have been combined by putting

$$T_{ij} = \sigma_{ij} - \rho u_i \, u_j \tag{8}$$

Applying Gauss' theorem we find, as before,

$$\frac{\partial}{\partial t}(\rho u_i \, b_i) = (T_{ij} \, b_i)_{,j} \tag{9}$$

which is similar to Eq. X.96 of Ref. 2. In rectangular coordinates this becomes, after combining with Eq. 6

$$\rho \frac{Du_i}{Dt} = \sigma_{ij,j} \tag{10}$$

5. CONSERVATION OF ENERGY

The energy equation is obtained by a process similar to those followed above. The total energy per unit mass is

$$E = I + \tfrac{1}{2}u_j \, u_j \tag{11}$$

where I denotes the internal energy per unit mass. Since the rate of change of the total energy within $\bar{V}$ must equal the rate at which work is done by the surface tractions we can write

$$\frac{D}{Dt}\int_{\bar{V}} \rho(I + \tfrac{1}{2}u_j\, u_j)\, dv = \int_{\bar{S}} \sigma_{ij}\, u_j\, n_i\, dS \tag{12}$$

Putting ρE for H in Eq. 1 we obtain the energy equation

$$\frac{\partial}{\partial t}\int_{\tilde{V}} \rho E\, dv = \int_{\tilde{S}} (\sigma_{ij} - \rho E\delta_{ij})\, u_i\, n_j\, dS \tag{13}$$

which, by the same arguments as before, leads to the result

$$\frac{\partial}{\partial t}(\rho E) = \frac{\partial}{\partial x_i}[\sigma_{ij}u_j - \rho E u_i] \tag{14}$$

This can be combined with the momentum equation to obtain

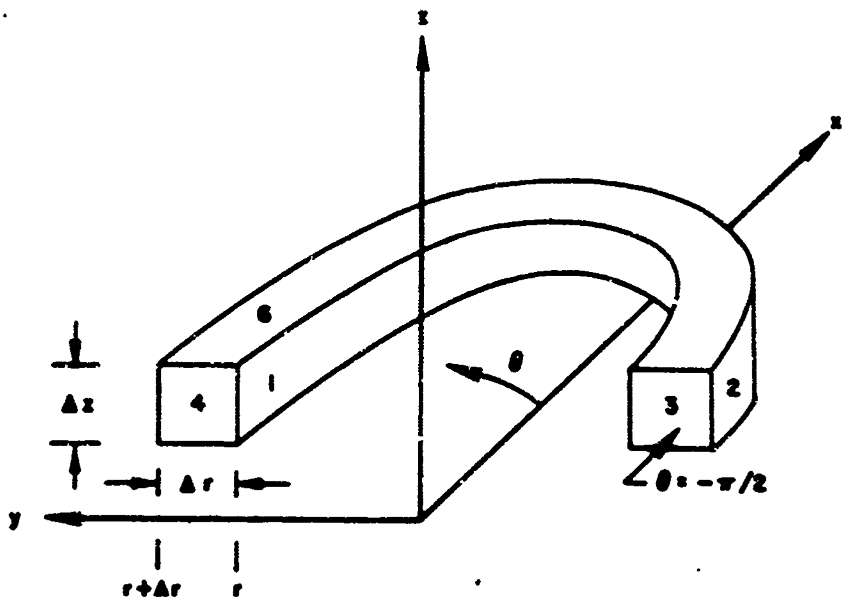
$$\rho\, \frac{DI}{Dt} = \sigma_{ij}\, u_{j,i} \tag{15}$$

which is just the first law of thermodynamics in hydrodynamic notation.

6. THE FINITE DIFFERENCE EQUATIONS

The integral relations of Eqs. 2, 7 and 13 can be used to write out directly the flow equations in finite difference form. This approach seems natural since it suggests a finite difference scheme in which the conservation f mass, momentum and energy are naturally assured, and a physical interpretation of each step in the numerical computation is possible. An appropriate control volume for flow with axial symmetry is sketched in Fig. 2. It consists of half a toroid of rectangular section. The choice of a half ring allows for a rigorous determination of the momentum equation for radial velocities, but for the other equations a complete toroid would do just as well.

If the integrands in Eq. 2 for mass conservation are taken to be constant and $\tilde{V}$ is the half-ring of Fig. 2, then the surface integral vanishes over S_3 and S_4 in view of the zero angular velocity, but the integral over the other four surfaces is finite.



Surface Number	Components of $\bar{n} = (n_1, n_2, n_3)$ parallel to		
	(r)	(θ)	(z)
1	-1	0	0
2	1	0	0
3	0	-1	0
4	0	1	0
5	0	0	-1
6	0	0	1

Fig. 2--The shape of an element of volume, $\tilde{V}$, appropriate for deriving the Eulerian finite difference equations in cylindrical coordinates is illustrated in the sketch.

Then Eq. 2 becomes

$$\dot{\rho}\,\tilde{V} = -\,2\pi\left\{\rho r u\,\Big|_{1}^{2}\,\Delta z + \rho v\,\Big|_{5}^{6}\,(r+\tfrac{1}{2}\Delta r)\,\Delta r\right\} \tag{16}$$

In this and subsequent equations we specialize the notation, u_i, in view of the cylindrical geometry, to

$$u_1 = u,\ u_2 = 0,\ u_3 = v \tag{17}$$

Eq. 16 is the mass transport equation used in the finite difference scheme, except for replacement of the time derivative by a finite difference.

The radial momentum equation is obtained by means of Eq. 7 in which b_i is the unit vector in the (fixed) x direction. Then, with θ the azimuth angle to the x-axis,

$$b_1 = \cos\theta,\ b_2 = -\sin\theta,\ b_3 = 0 \tag{18}$$

In view of the symmetry several components of T_{ij} vanish, viz.

$$(T_{ij}) = \begin{pmatrix} T_{11} & 0 & T_{13} \\ 0 & T_{22} & 0 \\ T_{31} & 0 & T_{33} \end{pmatrix} \tag{19}$$

and the integral equation is

$$\frac{\partial}{\partial t}\int_{\tilde{V}} \rho\, u\,\cos\theta\, dv = \int_{\tilde{S}} (T_{1j}\cos\theta - T_{2j}\sin\theta)\, n_j\, dS$$

As before, it is assumed that ρ and u are constant in $\tilde{V}$ and T_{ij} is constant on each of the faces of the cell. Carrying out the integrals, we find, upon multiplication by π,

$$\frac{\partial}{\partial t}(\rho u\,\tilde{V}) = 2\pi\left\{rT_{11}\,\Big|_{1}^{2}\,\Delta z - T_{22}\,\Delta r\,\Delta z + T_{13}\,\Big|_{5}^{6}\,\Delta r(r+\tfrac{1}{2}\Delta r)\right\} \tag{20}$$

For the momentum equation in the axial direction we put for b_i

$$b_1 = 0,\ b_2 = 0,\ b_3 = 1$$

which, with Eq. 7, leads to

$$\frac{\partial}{\partial t} (\rho v \tilde{V}) = 2\pi \left\{ rT_{31} \Big|_1^2 \Delta z + T_{33} \Big|_5^6 (r+\tfrac{1}{2}\Delta r) \Delta r \right\} \tag{21}$$

The finite-difference energy equation follows by carrying out the integrals of Eq. 13 over $\tilde{V}$, leading to

$$\frac{\partial}{\partial t} (E\tilde{V}) = 2\pi \left\{ r[\sigma_{11} - \rho E) u + \sigma_{31} v] \Big|_1^2 \Delta z \right.$$

$$\left. + [\sigma_{13} u + (\sigma_{33} - \rho E) v \Big|_5^6 (r+\tfrac{1}{2}\Delta r) \Delta r \right\} \quad . \tag{22}$$

In summary, we have four equations for the conservation of mass, two components of momentum and the energy. By means of Eq. 27 we define A_1 through A_6, and in addition we put m for $\rho\tilde{V}$ to obtain the equations in their most compact form. These equations are essentially those used in the computer program, except for the finite time step.

$$\frac{\partial m}{\partial t} = - \rho u A\Big|_1^2 - \rho v A\Big|_5^6 \tag{23}$$

$$\frac{\partial (mu)}{\partial t} = T_{11} A\Big|_1^2 - T_{22} A_4 + T_{13} A\Big|_5^6 \tag{24}$$

$$\frac{\partial (mv)}{\partial t} = T_{31} A\Big|_1^2 + T_{33} A\Big|_5^6 \tag{25}$$

$$\frac{\partial (mE)}{\partial t} = [(\sigma_{11} - \rho E) u + \sigma_{31} v] A\Big|_1^2$$

$$+ [\sigma_{13} u + (\sigma_{33} - \rho E) v] A\Big|_5^6 \tag{26}$$

The surfaces referred to are illustrated in Fig. 2, and the value of A on the $i^{\underline{th}}$ surface is denoted by A_i.

$$\left.\begin{aligned} A_1 &= 2\pi\, r\, \Delta z,\ A_2 = 2\pi\, (r+\Delta r)\, \Delta z \\ A_3 &= A_4 = 2\pi\, \Delta r\, \Delta z \\ A_5 &= A_6 = 2\pi\, (r+\tfrac{1}{2}\Delta r)\, \Delta r \end{aligned}\right\} \tag{27}$$

7. DIFFERENTIAL EQUATIONS FOR FLOW WITH CYLINDRICAL SYMMETRY

For completeness the flow equations in differential form which are obtained by passing to the limit of vanishing Δr and Δz are noted below.

$$\frac{\partial \rho}{\partial t} = - \frac{1}{r} \frac{\partial (\rho u r)}{\partial r} - \frac{\partial (\rho v)}{\partial z} \tag{28}$$

$$\frac{\partial (\rho u)}{\partial t} = \frac{1}{r} \frac{\partial (T_{11} r)}{\partial r} + \frac{\partial T_{13}}{\partial z} - \frac{T_{22}}{r} \tag{29}$$

$$\frac{\partial (\rho v)}{\partial t} = \frac{1}{r} \frac{\partial (T_{13} r)}{\partial r} + \frac{\partial T_{33}}{\partial z} \tag{30}$$

$$\begin{aligned} \frac{\partial (\rho E)}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left\{ [(\sigma_{11} - \rho E)\, u + \sigma_{31}\, v]\, r \right\} \\ &\quad + \frac{\partial}{\partial z} \left\{ \sigma_{13}\, u + (\sigma_{33} - \rho E)\, v \right\} \end{aligned} \tag{31}$$

8. ISOTROPIC AND DEVIATOR COMPONENTS OF STRESS AND STRAIN

The stresses σ_{ij} can be separated into two parts, a pressure (isotropic) component p, which is the average of the principal stresses, and a deviator component, S_{ij}, which results from subtracting out the pressure term from the total stress. Thus we can write

$$\sigma_{ij} = - p\, \delta_{ij} + S_{ij} \tag{32}$$

and it follows that

$$p = - \tfrac{1}{3} \sigma_{ii} \tag{33}$$

and

$$S_{ii} = 0 \tag{34}$$

The negative sign preceeding the pressure in Eq. 32 is required to conform with the usual conventions that pressure is positive for material in compression and stresses are positive for material in tension.

In a similar manner the velocity gradient can be resolved into a rotation, a compression and a distortion. To accomplish this the strain-rate tensor ϵ_{ij} is defined as the symmetric component of the gradient of the velocity field

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{35}$$

and the skew part

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \tag{36}$$

of the tensor $u_{i,j}$ defines the vorticity. The commas in these expressions denote differentiation in the sense of tensor analysis. Since the flows of interest in this report have axial symmetry, the matrix of strain-rates is noted below for that special case:

$$(\epsilon_{ij}) = \begin{Bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) \\ 0 & \frac{u}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) & 0 & \frac{\partial v}{\partial z} \end{Bmatrix} \tag{37}$$

The strain rate can, in turn, be broken up into an isotropic part involving the rate of change of density and a deviator part obtained by subtracting out this isotropic component. Specifically,

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\,\delta_{ij} \tag{38}$$

A relation between strain-rate and density is obtained by combining Eqs. 6 and 35:

$$\epsilon_{ii} = u_{i,i} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{V}\frac{DV}{Dt} \tag{39}$$

The pressure is determined by the equation of state, $p(E,\rho)$. A general discussion of current information on the equation of state is given by Brush[3] and a specific empirical fit to the existing data is described by Tillotson.[4]

To complete the description of the flow a specific relation between stress and strain is needed. A general form of such an equation is displayed and discussed below to place the particular equations used to formulate the OIL-RPM code in context. Subsequently, it will be specialized to describe the rigid-plastic model used in the computer program.

An equation of the form

$$\dot{e}_{ij} = \frac{1}{2\mu} \dot{S}_{ij} + b\, S_{ij} \tag{40}$$

relating strain-rate, stress-rate and stress includes the description of classical viscosity, elasticity, and the Prandtl-Reuss equations of plastic flow as special cases. Here b and μ are scalars, but not necessarily constants. The dots denote differentiation following a set of axes fixed in the material. Thus, special provision has to be made in the derivatives if there is rotation or translation of the material.

If $\mu = \infty$ and $b = 1/2\rho\nu$ the equations describe classical viscosity with ν the kinematic viscosity and $\rho\nu$ the dynamic viscosity of the material. The full equations are then the Navier-Stokes equations if the additional requirement is made that the density be held constant, as in incompressible flow.

Elasticity is described by putting $b = 0$, and in that case μ is the rigidity modulus of the material, which is one of the two Lamé constants. A second elastic constant is included in the equation of state of the material, which for moderate internal energy, I, and near-normal density is approximated by $p = A(\rho/\rho_0 - 1)$, where A is the bulk modulus in the notation of this report. The other Lamé coefficient is expressible in terms of A and μ by $\lambda = A - \frac{2}{3}\mu$.

The Prandtl-Reuss equations for an elastic-plastic medium with a von Mises yield criterion are discussed by Hill,[5] Thomas[6] and numerous other authors on plasticity. If, in addition to Eq. 40, we require the von Mises yield criterion

$$S_{ij}\, S_{ij} = 2Y^2 \tag{41}$$

where Y is the yield stress in pure shear, we obtain an explicit expression for b,

$$b = \frac{S_{ij}\, \dot{e}_{ij}}{2Y^2} \tag{42}$$

which completes the specification of the medium, except for the possibility that the deformation is elastic. In that case $b = 0$. At each step in the calculation the deformation has to be tested to see whether the new stress is elastic, i.e. if

$$S_{ij}\, S_{ij} < 2Y^2 \tag{43}$$

or plastic, which is the case when the inequality is violated, as in Eq. 41.

The full Prandtl-Reuss equations have to account for material rotation by means of additional terms which were originally described by Jaumann (see Prager[7] for a discussion), and are described by Thomas[6] on p. 88. In our notation

$$\dot{S}_{ij} = \partial_t\, S_{ij} + S_{kj}\, \omega_{ki} + S_{ik}\, \omega_{kj} \quad , \tag{44}$$

where ω_{ij} is given by Eq. 36 and ∂_t denotes an ordinary time derivative in fixed axes.

The rigid-plastic model constitutes an important simplification which is useful when the strains are large compared to the value at yield, as for the "near field" in hypervelocity impact. In this approximation the elastic portion $\dot{S}_{ij}/2\mu$ of the total strain is set to zero, resulting in the constitutive relation

$$\dot{e}_{ij} = b\, S_{ij} \tag{45}$$

This may be thought of as an approximation in which μ is very large. In this limit of an infinite modulus of rigidity it is said that the material is described by a "rigid-plastic" model. Setting μ equal to infinity eleminates the need for calculating the Jaumenn terms, in Eq. 44, and allows the stresses to be calculated directly from the strain-rate. To do this Eq. 45 is multiplied by itself, with the result

$$\dot{e}_{ij}\,\dot{e}_{ij} = b^2\, S_{ij}\, S_{ij}$$

Accounting for the von Mises condition, Eq. 41, we have

$$S_{ij} = \frac{\sqrt{2}\, Y}{\sqrt{\dot{e}_{kl}\,\dot{e}_{kl}}}\,\dot{e}_{ij} \tag{46}$$

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